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CONFORMAL MAPPINGS AND BOUNDARY VALUE PROBLEMS.(U)

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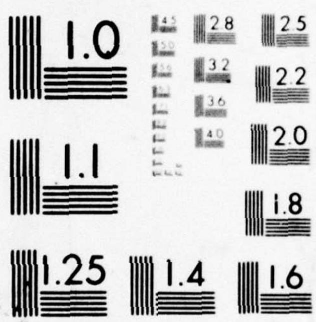

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CONFORMAL MAPPINGS AND BOUNDARY VALUE PROBLEMS

Final Technical Report

by

J. Burbea

A. Ghandour

R. Mandelbaum

OCTOBER, 1976

EUROPEAN RESEARCH OFFICE

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20. Abstract

Three principal areas of investigation have been followed.

1. Kernal functions and related areas.

Results have been obtained on polynomial density in Ber's Spaces, Berman Spaces over multiply-connected domains, Total Positivity and reproducing kernels, Szego kernels and the Riesz projection theorem and Metric on Annuli.

2. BVP and IVP.

Study has been undertaken of transforming BVP into IVP. In particular, a method whereby a well-posed elliptic boundary-value problem of the Dirichlet type is transformed into a first-order non-linear equation governing the Green's function of an embedded problem is studied.

3. Singularities.

The study of smoothings of analytic singularities is discussed. In particular, generalized complete intersections and their spaces of deformations are analyzed.

Brief Table of Contents

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Three principal areas of investigation ~~have been~~ are as follow~~ed~~s,

1. Kernel functions and related areas,

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3. Singularities.

The study of smoothings of analytic singularities is discussed. In particular, generalized complete intersections and their spaces of deformations are analyzed.

The objectives of research undertaken under Research Contract No. DA-ERO-124-74-G0064 were the following:

(1) Determination of moduli of multiply connected plane domains.

(2) Computation of conformal mappings onto canonical domains.

(3) Variations of kernel functions with respect to its domain.

(4) Transformation of boundary value problems into initial value problems.

(5) Analogues of Schwarz-Pick lemma, distortion theorems.

(6) Numerical applications to the theory of elasticity and fluid dynamics.

I. Work on Kernel Functions and Related Areas.

Work has been done in the area of kernel functions in plane domains and metrics in plane domains. The following results have been obtained:

1. Polynomial Density in Bers spaces I.

Let D be a bounded simply connected domain such that $\iint_D \lambda_D^{2-q} dxdy < \infty$ for $q > 1$. Here $\lambda_D(z)$ is the Poincaré metric for D. Define $A_q^p(D)$, the Bers space, to be the Frechét space of holomorphic functions f on D, such that $\|f\|_{q,p}^p = \iint_D \lambda_D^{2-q} |f|^p dxdy$ is finite, $0 < p < \infty$, $qp > 1$. It is well known that the polynomials are dense in $A_q^p(D)$ for $qp \geq 2$. We show that they are dense in $A_q^p(D)$ for $qp > 1$ irrespective whether the boundary of D is rectifiable or not. Accepted for publication in the "Proceeding of the Amer. Math. Society" (Feb. 23, 1976).

2. Polynomial Density in Bers spaces II.

This paper is a continuation and a generalization of the previous work (item 1). Here we assume that

$$t_D = \sup \{q \in R: \mu_q(D) = \infty\}, \quad \mu_q(D) = \iint_D \lambda_D^{2-q} dxdy$$

and so $1 \leq t_D \leq 2$. We let

$$Q(t_D) = \begin{cases} [t_D, \infty), & \mu_{t_D}(D) < \infty \\ (t_D, \infty), & \mu_{t_D}(D) = \infty \end{cases}$$

and note that $\{q \in R: \mu_q(D) < \infty\} = Q(t_D)$. Of course $Q(1) = (1, \infty)$ and $Q(2) = [2, \infty)$. With the notation as above we show that

Theorem 1. The polynomial are dense in $A_q^p(D)$, $0 < p < \infty$, $qp \in Q(t_D)$.

Theorem 2. The following hold

(i) $1 \le t_D \le 2$.

(ii) If D is a Jordan domain with a rectifiable boundary ∂D then $t_D = 1$.

(iii) There is a Jordan domain with a non rectifiable boundary with $t_D = 1$.

(iv) There is a domain D with $1.17 < t_D < 2$.

(v) There is a domain D with $t_D = 2$.

submitted to the "J. of Lond. Math. Soc."

3. Profjection on Bergman Spaces over Multiply Connected Domains.

Let D be a bounded domain of finite connectivity (with some smoothness requirements on its boundary). The Bergman space of D, $B_p(D)$ is the set of all functions $f(z)$, analytic in D, for which $\|f\|_p = \{\iint_D |f(z)|^p d\omega(z)\}^{1/p} < \infty$, $1 \le p < \infty$. Here $d\omega(z) = dxdy$. The "natural projection of $L_p(D)$ to $B_p(D)$ is given by $(Pf)(\zeta) = \iint_D f(z)K_D(\zeta,\bar{z})d\omega(z)$, where $K_D(z,\bar{\zeta})$ is the Bergman kernel for D. If $p = 1$, this projection is not bounded. The Ahlfors-Bers theory does not seem to help in case $1 < p < \infty$. Here we show that P is a bounded projection of $L_p(D)$ onto $B_p(D)$, $1 < p < \infty$. Moreover, the dual of $B_p(D)$ is isomorphic to $B_{p'}(D)$, $1/p + 1/p' = 1$, $1 < p < \infty$. For the special case when D is the unit disc these results were obtained by various authors; e.g., Zaharjuta and Judovič; Shields and Williams, and Forelli and Rudin. Submitted to the "J. für die reine und angewandte Mathematic"

4. Total Positivity and Reproducing Kernels.

Here we investigate the relationship between total positivity and reproducing kernels. We extend the notion of total positivity to domain in the complex plane. In doing so, we also give a geometrical interpretation to certain

Wronskians of reproducing kernels. Appeared in the "Pacific J. of Math. Vol. 55 (1974), 343-359.

5. <u>Total Positivity of Certain Reproducing Kernels.</u>

Here we study the total positivity of various kernels, especially reproducing kernels of Hilbert spaces of analytic functions. We do so by employing a familiar device known as the "Composition formula of Polya and Szego". Using this formula we are able to give a short proof for the variational diminishing property of a generalized analogue of the la Vallee Poussin menas. This generalizes earlier work of Polya and Schoenberg and recent work of Horton. Our method is based on the isometrical image of the reproducing kernel called the generating function. The reproducing kernel is then expressed as a composition of two generating functions so that the problem is reduced to investigating the total positivity of the generating function. This method extends earlier work and yields many new reproducing kernels which are total positive. Submitted to the "Pacific J. of Math."

6. <u>Additional Current Work.</u>

a.

Jacob Burbea, The Pennsylvania State University, University Park, Pa. 16802. <u>The Szegö kernel and the Riesz's projection theorem.</u>

et D be a plane domain whose boundary consists of a finite number of disjoint analytic curves. This restriction could be weakened considerably). The Hardy-Szegö space $H_p(D)$ is regarded as a losed subspace of $L_p(\partial D)$, $1 \le p \le \infty$, in the usual way. Let $K_D(z,\bar{\zeta})$ be the Szegö kernel for D a et $(Pf)(\zeta) = \int_{\partial D} K_D(\zeta,\bar{z})f(z)|dz|$ be the "natural projection" of $L_p(\partial D)$ to $H_p(D)$. <u>Theorem</u>. P continuous projection from $L_p(\partial D)$ onto $H_p(D)$, $1 \le p \le \infty$, and $A_p^{(2)} \le \|P\|_p \le A_p^{(1)}$ where $A_p^{(j)} =$ $^{(j)}/p-1 + c^{(j)}p$. Here $k^{(j)}$, $c^{(j)}$ $(j=1,2)$ depend only on D. When D is the unit disc this theore the classical Riesz's projection theorem. <u>Corollary</u>. For $L_p(\partial D) = H_p(D) \oplus H_q^{\perp}(D)$, $1/p + 1/q = 1$ e have $H_q^{\perp}(D) = \overline{z' H_p(D)}$. Here $z' = z'(s)$ where $z = z(s)$ is the parametrization of ∂D with espect to the length parameter s.

b. Jacob Burbea, The Pennsylvania State University, University Park, Pa. 16802 6.
Metrics on an annulus.

Let R be the annulus $\{z:r < |z| < 1\}$, $0 < r < 1$. The Carathéodory metric for R is given by

$$(*) \qquad dC_R^2(z) = |z|^{-2} \{\wp(2 \log |z|:\omega_1,\omega_2)-e_3\}|dz|^2$$

where $\wp$ is the Weierstrass $\wp$-function with the half periods $\omega_1 = \pi i$ and $\omega_2 = \log r$. Here $e_3 = \wp(\omega_1+\omega_2:\omega_1,\omega_2)$. Formula (*) settles a question raised in Kobayashi ["Hyperbolic Manifolds and Holomorphic Mappings", Marcel Dekker, New York, 1970], p. 52. It is well known that $dC_R^2(z) \le dP_R^2(z)$, where $dP_R^2(z)$ is the Poincaré metric for R. In this case $dP_R^2(z)$ is also the Kobayashi metric for R. We show that $dC_R^2(z) < dP_R^2(z)$. This could be regarded as an example in which for non symmetric domain the Kobayashi metric could be strictly bigger than the Carathéodory metric for each point of the domain.

c. The Carathéodory metric in plane domains.

Let D be a domain in $C \cup \{\infty\}$ whose boundary consists of more than two points and let $dP_D^2(z)$ be its Poincaré metric. Theorem 1. Let D as before and let $dS_g^2(z) = g(z,\bar{z})|dz|^2$ be a conformally invariant Kaehler metric on D. (i) If D is simply connected the curvature of dS_g^2 is constant and if also dS_g^2 is complete this constant is negative. (ii) If dS_g^2 is complete and $g(z,\bar{z})$ is a single-valued analytic function in $(z,\bar{z})$ then if the curvature is contant (must be negative) D is simply connected. The Carathéodory metric $dC_D^2(z)$ is given by $dC_D^2(z) = [2\pi K_D(z,\bar{z})]^2|dz|^2$ where $K_D(z,\bar{z})$ is the Szegö kernel for D. Theorem 2. $dC_D^2(z)$ has a negative curvature whose value on ∂D is -4. According to Theorem 1 the curvature is not a constant unless D is simply connected. It is well known that $dC_D^2(z) \le dP_D^2(z)$, in fact we show that $dC_D^2(z) < dP_D^2(z)$ when D is doubly connected.

d. The dual of the Bergman space in plane domains.

Let D be a bounded plane domain. The Bergman space of D $A_p(D)$ is the set of all functions $f(z)$, analytic in D for which $||f||_p = \{\int_D\int |f(z)|^p dxdy\}^{1/p} < \infty$, $1 \leq p < \infty$. The "natural projection" of $L_p(D)$ to $A_p(D)$ is defined by means of the Bergman kernel for D. If $p = 1$, this projection is not continuous. The theory of Bers does not seem to help in case $1 < p < \infty$. Theorem. P is continuous from $L_p(D)$ onto $A_p(D)$, $1 < p < \infty$ and $C_p^{(2)} \leq ||P||_p \leq C_p^{(1)}$, where $C_p^{(j)} = k^{(j)}/p-1 + c^{(j)}p$, $k^{(j)}$, $c^{(j)}$ are constants depending only on D. When D is the unit disc this theorem was first proved by Zaharjuta and Judovic [Uspehi Mat. Nauk., 19(1964), No. 2(116), 139-192]. Theorem 2. The dual of $A_p(D)$ is isomorphic to $A_q(D)$, $1 < p < \infty$, $1/p + 1/q = 1$. Unless $p = 2$, $A_p^*(D)$ is not isometric to $A_q(D)$ and the "isometry distortion" I_q satisfies $C_q^{(2)} \leq I_q \leq C_q^{(1)}$.

e. The annihilator of the Bergman space in plane domain.

Let $A_p(D)$, $1 < p < \infty$ be the Bergman space in a bounded plane domain D. The natural projection involving the Bergman kernel) P is continuous and so $L_p(D) = A_p(D) \oplus A_q^{\perp}(D)$ where $A_q^{\perp}(D)$ is the annihilator of $A_q(D)$, $1/p + 1/q = 1$. Theorem 1. $A_q^{\perp}(D) = \{i\frac{\partial h}{\partial z}: h, \frac{\partial h}{\partial z} \in L_p(D)$ and h "vanishes on ∂D"$\}$. This generalizes a result of Schiffer [Rend. Mat. e Appl. 22(1963). 447-468] when $p = 2$. Assume D has a Green function $G_D(z,\zeta)$ and define $(Lf)(\zeta) = -\frac{2}{\pi}\int_D\int \frac{\partial^2 G_D}{\partial z \partial \zeta} \overline{f(z)}dxdy$. This operator is in

fact a Hilbert transform. <u>Theorem 2</u>. Let $f \in L_p(D)$, $1 < p < \infty$. The $L^2 f = (I-P)f$. This generalizes a result of Block [Proc. Amer. Math. Soc. 4(1953), 110-117]. The above theorems enable us to deduce various projection theorems on operators defined by means of domain functions.

II. Work on Boundary Value Problems.

With reference to the problem of transforming boundary value problems to Initial Value Problems, the following results have been obtained:

1. Introduction:

In this note we present formally a method whereby a well posed elliptic boundary value problem of the Dirichlet type is transformed into seeking a solution for a first order non-linear equation governing the Green's function of an embedded problem. We examine an operator defined on a domain bounded by an analytic closed curve, and exploiting properties of the mapping function defined in terms of the kernel function, we map the domain into a disk On this new domain we examine the variations of the Green's function with changes in the domain. The method can be extended to multiply connected domains and to Neumann type problems.

2. The Formulation:

Suppose D is a bounded simply connected domain in $\mathbb{C}$ with a boundary curve C of class C^n $(n>2)$. Let $t_o \in D$. Let $f(z,t_o)$ be the unique analytic mapping which maps D onto $\Delta_{R_o} = \{w \mid |w| < R_o\}$ and such that

$$f(t_o,t_o) = 0 \quad ; \quad \frac{\partial f}{\partial z}(t_o,t_o) = 1 \qquad (1)$$

Suppose C_R is the curve $f^{-1}(\partial\Delta_R)$, $R \le R_o$, bounding the simply connected domain $D_R \subset D$. Then, the family of curves and domains (C_R, D_R) have the following properties:

a) $C_{R_o} = C$; $D_{R_o} = D$ and $C_o = D_o = t_o$

b) $C_R \cap C_{R'} = \phi$ if $R \neq R'$

c) $D_{R'} \subset D_R$ if $R' < R$

To the domain D is associated the kernel function [H]

$$K(z,t_o) = \frac{1}{\pi R_o^2} f'(z,t_o) \quad \text{with} \quad z \varepsilon D \tag{2}$$

and to the domains D_R we associate the kernel function

$$K_R(z,t_o) = \frac{1}{\pi R^2} f'(z,t_o) \tag{3}$$

In (3), f is understood to be the restriction of the mapping function associated with the domain D to the domain D_R.

Consider the elliptic operator

$$L = \sum_{i,j}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b_i \frac{\partial}{\partial x_i} - d \tag{4}$$

defined on D and which is assumed to be uniformly strongly elliptic and self-adjoint with the coefficients a_{ij}, b_i and d functions of x defined on D and sufficiently smooth such that a unique solution exists to the Dirichlet problem [F].

$$Lu = 0, \quad u = 0 \quad \text{on} \quad C \tag{5}$$

We denote the restriction of the operator L to D_R by L_R. There then exists for <u>each</u> domain D_R a Green's function (the fundamental solution) $G_R(p_1,p_2)$ associated with $L_R u_R = F$, $u_R = 0$ on C_R, whose solution is simply given by

$$u_R(p_1) = \int_{D_R} F(p_2)\, G_R(p_1,p_2)\, dA \tag{6}$$

Denote by Δ_R the disk of radius R . By utilizing the mapping function f in L_R we can get a new operator $\hat{L}_R$ defined on Δ_R , namely,

$$\hat{L}_R = \sum_{i,j}^{2} \hat{a}_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{2} \hat{b}_i \frac{\partial}{\partial y_i} - \hat{d} \tag{7}$$

where now $(y_1, y_2) = (r, \theta)$ and the coefficients are functions of the new variables. We now normalize our coordinates by means of the coordinate transformation $\theta = \theta$ and $r = \rho R$ to get the new operator $\tilde{L}_R$ instead of (7) with $\hat{a}_{ij} \to \tilde{a}_{ij}$, $\hat{b}_i \to \tilde{b}_i$ and $\hat{d} \to \tilde{d}$ now functions of (ρ, θ) and the parameter R .

For simplification of the analysis, now specialize to the case where the coefficient $\tilde{d}$ in $\tilde{L}$ is such that $\tilde{d}(\rho,\theta) = R^2 \hat{d}(\rho R, \theta) > 0$ and $\tilde{a}_{ij}$ and $\tilde{b}_i$ are independent of $\tilde{R}$. Define the following operator: $\tilde{M}_R = \tilde{L}_R + R^2 \hat{d}$, which is independent of R . By the fundamental properties of the Green's function, $\tilde{G}_R$ associated with $\tilde{L}_R$ and the use of Green's theorem and by examining the dependence of $\tilde{G}_R$ on the parameter R we can derive the following initial value problem for the Green's function:

$$\frac{\partial G(P,Q)}{\partial R} - \int_{\Delta} \left[R^2 \rho \frac{\partial \hat{d}}{\partial R} + 2R\hat{d} \right] \tilde{G}_R(P,t)\, \tilde{G}_R(t,Q)\, dA(t) \tag{8}$$

where $t = (\rho, \theta)$ and with (8) is associated the initial condition

$$\tilde{G}_R(P,Q) \Big|_{R=0} = \tilde{G}_o(P,Q) \tag{9}$$

where $\tilde{G}_o(P,Q)$ is the unique function such that $\tilde{M}_R \tilde{G}_o = -\delta(P-Q)$ with $\tilde{G}_o$ vanishing for $P \in \partial\Delta_R$.

Equation (8) constitutes the basic new problem which need be solved. In general, for this specialized case, for sufficiently complicated $\hat{d}$ one has to tackle the problem by numerical techniques and iterative procedures. The formulation above has assumed the existence of a kernel function which mapped the domain D_R onto the disk Δ_R. Following the methods outlined in [2], one can find the kernel function utilizing appropriate numerical schemes.

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Friedman, A., Partial Differential Equations, Holt & Rinehart, 1971.

Hille, E., Analytic Function Theory, Vol. II, Ginn & Co., 1962.

III. Work on Distortion Theorems.

As an offshoot of the work on distortion theorems and moduli of domains, further investigations into the general theory of deformations were begun (see Status Report #3).

The following preliminary results have been obtained in this area:

SMOOTHING PERFECT VARIETIES

R. MANDELBAUM* AND M. SCHAPS

0. Introduction. In this research report we discuss the deformation theory of intersections of germs of perfect analytic varieties. It is well known that hypersurface singularities are always smoothable and that the parameter space S of the versal deformation space of a hypersurface singularity is isomorphic to the parameter space of the space of infinitesimal first-order deformations of the given hypersuface. As noted in [5] the same results are true for complete intersections of hypersurfaces. If we move on from hypersurfaces to pure codimension two analytic objects and in addition add the hypothesis of perfectness we find similar phenomena occurring. In particular in [9], [11] it is shown that a germ X of a perfect analytic variety of codimension two in C^n ($n \leq 5$) will always be smoothable and if $n > 5$ then even though X is not generically smoothable it nevertheless has a well-understood generic form X' whose singular locus $\mathscr{S}(X')$ has codimension 4 in X'. In Theorem 1 we show that a proper intersection $X = \bigcap X_i$, of perfect germs of analytic varieties has smoothness properties at least as good as those of the individual germs X_i. Thus if the X_i all have codimension at most two then X will always be deformable to a germ X' with $\operatorname{codim}(\mathscr{S}(X'), X') \geq 4$. In particular if $\dim X \leq 3$ it will always be smoothable.

In [9], [11] it is also shown that all first-order deformations of germs of perfect analytic varieties of codimension two in C^n can be lifted unobstructedly to flat analytic deformations of the germs. In Theorem 2 we show that the same is true for proper intersections of such germs.

We deal throughout with germs of analytic subvarieties at the origin in some C^n as defined, for example, in [3], [4]. $\mathcal{O}_X$ will denote the structure sheaf of the subvariety X, $\mathscr{I}(X)$ its defining ideal and its singular locus. All other definitions and notation will be as in [3], [4].

AMS (MOS) subject classifications (1970). Primary

*The first author was supported in part by the European Research Office under contract DAERO-12474-G0069 during the period when this research was conducted.

one isolated singular point, but not smoothable.

2. Smoothing intersections. To determine to what extent smoothability is preserved under intersection we first need some preliminary lemmas.

LEMMA 2.1 (CF. [7]). *Let P be a Noetherian local ring and suppose J is an ideal in P such that $B = P/J$ has projective dimension m as a P-module. Let N be a finite P-module.*

Then for all $i > m -$ depth $_J N$, $\operatorname{Tor}_i(B, N) = 0$.

PROOF. Induction on depth $_J N$.

LEMMA 2.2. *Suppose X_1, X_2 are perfect germs of analytic subvarieties at the origin in C^n. Let $\mathscr{I}_1 = \mathscr{I}(X_1)$, $\mathscr{I}_2 = \mathscr{I}(X_2)$, $\mathcal{O}_1 = {}_n\mathcal{O}/\mathscr{I}_1$ and $\mathcal{O}_2 = {}_n\mathcal{O}/\mathscr{I}_2$. Then, if* $\operatorname{ht}(\mathscr{I}_1 + \mathscr{I}_2) = \operatorname{ht}\mathscr{I}_1 + \operatorname{ht}\mathscr{I}_2$,

(1) $\operatorname{Tor}_i^{\mathcal{O}}(\mathcal{O}_1, \mathcal{O}_2) = 0$ *for $i > 0$,*

(2) $\mathscr{I}_1\mathscr{I}_2 = \mathscr{I}_1 \cap \mathscr{I}_2$,

(3) *$X_1 \cap X_2$ is a perfect germ.*

PROOF. (1) $\operatorname{ht}(\mathscr{I}_1 + \mathscr{I}_2) = \operatorname{ht}\mathscr{I}_1 + \operatorname{ht}\mathscr{I}_2$ implies depth $_{\mathscr{I}_1}\mathcal{O}_0 =$ depth $_{\mathscr{I}_1}\mathcal{O}_2$. Then, by lemma 1, $\operatorname{Tor}_i(\mathcal{O}_1, \mathcal{O}_2) = 0$ for $i <$ proj dim $_n\mathcal{O}_1 -$ depth $_{\mathscr{I}_1}\mathcal{O}^2$ $=$ depth $_{\mathscr{I}_1 n}\mathcal{O}_0 -$ depth $_{\mathscr{I}_1}\mathcal{O}_2 = 0$ since is perfect.

(2) Since $\mathscr{I}_1\mathscr{I}_2 \subset \mathscr{I}_1 \cap \mathscr{I}_2$ it suffices to show $\mathscr{I}_1 \cap \mathscr{I}_2 \subset \mathscr{I}_1\mathscr{I}_2$. Let

$$_n\mathcal{O}^m \xrightarrow{d} {}_n\mathcal{O}^r \xrightarrow{d_1} {}_n\mathcal{O} \xrightarrow{\varepsilon} \mathcal{O}_1 \to 0$$

be a segment of a free resolution of $\mathcal{O}_1$ obtained by setting $d_1(a) = a \cdot f$ where $f = (f_1, \cdots, f_r)$ and $\{f_1, \cdots, f_r\}$ generate $\mathscr{I}_1$. Then tensoring by $\mathcal{O}_2$ and using $\operatorname{Tor}_1(\mathcal{O}_1, \mathcal{O}_2)$ gives the desired result.

(3) is a straightforward calculation showing codim $(X_1 \cap X_2, C^n)$ $<$ proj dim $_n X_1$.

LEMMA 2.3. *Suppose X_1, X_2 are germs of analytic subvarieties at $0 \in C^n$ with $\mathscr{I}_1 = \mathscr{I}(X_1)$ and $\mathscr{I}_2 = \mathscr{I}(X_2)$ and suppose $\mathscr{I}_1 \cap \mathscr{I}_2 = \mathscr{I}_1 \cap \mathscr{I}_2$. Suppose $\mathscr{V}_1 = (V_1, \pi_1, T)$, $\mathscr{V}_2 = (V_2, \pi_2, T)$ are flat deformations of X_1, X_2 in C^n. Let $V = V_1 \cap V_2$, considered a subvariety in $C^n \times T$ with projection $\pi : \tilde{V} \to T$, and set $\tilde{\mathscr{V}} = (V, \pi, T)$. Then $\tilde{\mathscr{V}}$ is a flat deformation of $X_1 \cap X_2$ in C^n.*

PROOF. To show $\pi : \tilde{V} \to T$ is flat we must show that all the relations on $\mathscr{I}_1 + \mathscr{I}_2$ lift. We can demonstrate that since $\mathscr{I}_1 \cap \mathscr{I}_2 = \mathscr{I}_1\mathscr{I}_2$ all such relations are generated by relations on $\mathscr{I}_1$ and $\mathscr{I}_2$ and by trivial relations. But all such relations lift, so π is flat.

THEOREM I (CF. [8]). *Let X_1, X_2 be germs of perfect analytic subvarieties of C^n smoothable to order k. Suppose* $\operatorname{codim}(X_1 \cap X_2, C^n) = \operatorname{codim}(X_1, C^n) + \operatorname{codim}(X_2, C^n)$. *Then $X = X_1 \cap X_2$ is a germ of a perfect analytic subvariety of C^n smoothable to order k.*

PROOF. Let $\mathscr{V}_i = (V_i, \pi_i, T_i)$ be the hypothesized smoothing of X_i. Let $G = GA(n, C)$ be the affine transformations of C^n and let $T = G \times T_1 \times T_2$. Let $\tilde{\mathscr{V}}_1$ be the deformation of X_1 over T given by $\tilde{V}_{1,(g,t_1,t_2)} = g(V_{1,t_1})$ and $\tilde{\mathscr{V}}_2$ the deformation of X_2 given by $\tilde{V}_{2,(g,t_1,t_2)} = V_{2,t_2}$. Let $\mathscr{I} = \tilde{V}_1 \cap \tilde{V}_2$ in $C^n \times T$ and $\tilde{\mathscr{V}}$ be the corresponding deformation of X. Then $\mathscr{V}$ is a flat deformation Lemmas 2, 3 and it

1. Definitions. We recall that $\mathscr{V} = (V, \pi, T)$ is a flat deformation of X in Y if $\pi : V \to T$ is a flat map of germs of analytic varieties, X, Y are germs of analytic varieties, X a subvariety of Y, V a subvariety of $Y \times T$, and $X \Rightarrow V_0$. We can assume without loss of generality that $V_0 = X$ is defined in some open neighborhood of the origin in $Y = C^n$ by holomorphic equations $f_i(x) = 0$, $i = 1, \cdots, m$, and that V has equations $f_i(x, t) = 0$, $i = 1, \cdots, m$, in $C^n \times T$ with $f_i(x, 0) = f_i(x)$. As a working definition of flatness we take $\pi : V \to T$ is flat if every relation $r(x) = (r_1(x), \cdots, r_m(x))$ between the $f_1(x), \ldots, f_m(x)$ (i.e., $\sum r_i(x) f_i(x) = 0$) can be lifted to a relation $r(x, t) = (r_1(x, t), \cdots, r_m(x, t))$ between the $f_i(x, t)$.

If $\mathscr{V}$ is a flat deformation of X in C^n we shall say $\mathscr{V}$ is a smoothing of X to order k if the generic fiber V_t of $\mathscr{V}$ has singular locus Σ_t with codim $(\Sigma_t, V_t) \geqq k$. If V_t is nonsingular then $\mathscr{V}$ is a smoothing of X. We say X is smoothable to order k if it has a smoothing of this order ($k = \infty$ if and only if X is smoothable).

We call X rigid if all flat deformations of X are locally trivial. In particular a germ of a rigid singular variety X is not smoothable. Even nonrigid X may not be smoothable as shown by examples of Mumford and Schlessinger [10], [12]. In particular there exist curves in P^n which are not smoothable. On the other hand all analytic curves in C^3 are smoothable. [The question for reduced irreducible curves in P^3 is still open.]

We recall that given any germ of a k-dimensional variety (at the there exists a finite-analytic mapping $f: V \to C^k$ exhibiting ${}_V\mathcal{O}$ as a finitely generated ${}_k\mathcal{O}$-module. We say V is perfect if ${}_V\mathcal{O}$ is free as a ${}_k\mathcal{O}$-module.

This is of course equivalent to the Cohen-Macaulay condition that depth ${}_n\mathcal{O}({}_V\mathcal{O}) = \dim {}_V\mathcal{O} = \dim V$ where V is a subvariety of C^n. Now by [9], [11] if $V \subset C^n$ is a perfect germ of codimension 2 and $n \leqq 5$ then V is smoothable. Since all pure 1-dimensional varieties are perfect we find that all curves in C^3 are smoothable. The above results are in a sense best possible. If $n = 6$ the familiar example of the cone of the Segre embedding X of $P^1 \times P^2$ in P^6 is perfect of codimension 2 but of course not smoothable.

The key aspect of the proof of the above results of Schaps, Loday is showing that a germ of a perfect subvariety of codimension 2 is necessarily determinantal. (We shall that a germ of a variety V is determinantal of type (m, n, l) if $\mathscr{I}(V) \subset {}_N\mathcal{O}_0$ is generated by the $l \times l$ minors of some $m \times n$ matrix R with coefficients in ${}_N\mathcal{O}_0$ and ht $\mathscr{I}$ = codim $V = (m - l + 1)(n - l + 1)$. In particular if V is perfect of codimension 2 then $\mathscr{I}(V)$ is generated by the maximal minors of an $n \times (n - 1)$ matrix. Now it can be shown that if V is determinantal of type (m, n, l) then generically its singular locus will have codimension $(m - l + 2)(n - l + 2)$ and thus codim$(\mathscr{S}(V), V) = m + n - 2l + 3$.

Perfect subvarieties of codim 2 are determinantal of type $(n, n - 1, n - 1)$, codimension $(\mathscr{S}(V), V) = 4$, thus giving us the Schaps, Loday result. This also furnishes us with examples of perfect codim 2 varieties which are smoothable to order k, but not $k + 1$. The variety given by the 2×2 minors of

$$\begin{bmatrix} x_1 & x_4 \\ x_1x_2 & x_1x_5 \\ x_3 & x_6 \end{bmatrix}$$

will have singular locus of codimension one, will be smoothable to a variety with

$\to \mathrm{Hom}_{\mathcal{O}_{C^n}}(\mathcal{I}(X_i), \mathcal{O}_X)$ be the map $\alpha_i(T \otimes f)(g) = fT(g)$ for $f \in \mathcal{O}_X$, $g \in \mathcal{I}(X_i)$, $T \in N_{X_i}$. We define $N_{X,X_1} = \mathrm{Im}\, \alpha_1$; $N_{X,X_2} = \mathrm{Im}\, \alpha_2$.

The following can then be proven:

Lemma 3.1. *Let X_1, X_2 be perfect germs of analytic subvarieties at the origin in C^n which we assume to be defined by ideals J_1, J_2 respectively. Let $X = X_1 \cap X_2$ and suppose* $\mathrm{codim}(X, C^n) = \mathrm{codim}(X_1, C^n) + \mathrm{codim}(X_2, C^n)$. *Then if X_i, $i = 1, 2$, is either a complete intersection or of codimension 2, then* (1) α_i is onto, $i = 1, 2$, and (2) $N_X = N_{X,X_1} \oplus N_{X,X_2}$.

Using induction we then obtain

Corollary 3.2. Let $X_1, \cdots, X_r$ be a very proper sequence of perfect germs of analytic subvarieties at the origin in C^n and suppose each X_i is either a complete intersection or of codimension 2. Let $Y_i = X_1 \cap \cdots \cap X_{i-1} \cap X_{i+1} \cap \cdots \cap X_r$. *Then if $X = \bigcap_{i=1}^r X_i$ we have $N_X = \bigoplus_{i=1}^r N_{X,Y_i}$.*

We now state:

Theorem 2. *Let $X_1, \cdots, X_r$ be a very proper sequence of perfect germs of analytic subvarieties at the origin in C^n with $X = \bigcap_{i=1}^r X_i$. Suppose each X_i is either a complete intersection or of codimension 2. Then every element of T_X^1 lifts to a flat analytic deformation of X.*

Proof. Let $g \in N_X$ represent $[g] \in T_X^1$. Then by Corollary 3.2 we have $g = \bigoplus g_i$, $g_i \in N_{X,Y_i}$, and by definition $g_i = \alpha_i(h_i \otimes 1)$ for some infinitesimal deformation h_i of X_i. By [6], [9], h_i lifts to a flat analytic deformation H_i of X_i and, by Lemma 2.3, $\bigcap H_i$ is then a flat analytic deformation of X inducing g.

We now clearly have

Corollary 3.3. *Let $X, X_1, \cdots, X_r$ be as in Theorem 2. Suppose* $\dim_C T_X^1 = N < \infty$ *so that, by* [2] *X has an analytic versal deformation space $V \to S$. Then $S \approx C^N$.*

Remark. N_{X,Y_i} in the above theorem and corollary consists of the space of all infinitesimal first-order deformations of X obtained by holding Y_i fixed and moving only X_i. Thus by Corollary 3.2 and Theorem 2 every deformation of X can be written as a combination (intersection) of movements of X in Y_i obtained by holding Y_i fixed and moving only X_i. Note that even movements of X_i which are trivial deformations may induce nontrivial deformations of X. For example let X_1 be the perfect analytic subvariety of codimension two in C^4 given by the vanishing of the maximal minors of the relations matrix.

$$R = \begin{bmatrix} z_1 & z_2^2 \\ z_2 & z_3 \\ z_3^2 & z_4 \end{bmatrix}.$$

Let X_2 be the nonsingular hypersurface with defining equation $h = z_2^2 - z_3^2 + z_1 + z_4$. The deformation space $\tilde{X}$ of X is then given by intersecting the variety $\tilde{X}_1$ in $C^4(z_1, \cdots, z_4) \times C^{10}(t_1, \cdots, t_{10})$ defined by the relation matrix

$$\begin{bmatrix} z_1 & z_2^2 + t_3 z_2 + t_4 \\ z_2 & z_3 \\ z_3^2 + t_1 z_3 + t_2 & z_4 \end{bmatrix}$$

thus remains to show that the generic fiber V_t of V has singular locus Σ_t with codim $(\Sigma_t, V) \geq k$. Let $c_1 = \operatorname{codim}(X_1, C^n)$, $c_2 = \operatorname{codim}(X_2, C^n)$, $\Sigma_{1,s} = \mathscr{S}(V_{1,s})$, and $\Sigma_{2,t} = \mathscr{S}(V_2, t)$. Let $P: C^n \times T \to T_1 \times T_2$ be the canonical projection, $Z_{s,t} = P^{-1}(s, t)$ for $(s, t) \in T_1 \times T_2$, $\tilde{V}_{s,t} = \tilde{V} \cap Z_{s,t}$ and $p: \tilde{V}_{s,t} \to G$ the obvious projection. Define $F: Z_{s,t} \to C^n \times C^n$ by $F(z, g) = (g^{-1}(z), z)$ so that $\tilde{V}_{s,t} = F^{-1}(V_{1,s} \times V_{2,t})$. Let $\Sigma_{s,t} = \mathscr{S}(\tilde{V}_{s,t})$ and $\tilde{\Sigma}_{s,t} = F^{-1}(\Sigma_{1,s} \times V_{2,t}) \cup F^{-1}(V_{1,s} \times \Sigma)$.

Then $\tilde{\Sigma}_{s,t} \subset \Sigma_{s,t}$ and it can be shown that, for generic g, $\Sigma_{s,t} \cap p^{-1}(g) = \tilde{\Sigma}_{s,t} \cap p^{-1}(g)$. Now since F is flat [1], we obtain

$$\operatorname{codim}(\tilde{\Sigma}_{s,t}; \tilde{V}_{s,t}) \geqq \min(\operatorname{codim}(\Sigma_{1,s}; V_{1,s}), \operatorname{codim}(\Sigma_{2,t}; V_{2,t})) \geqq k,$$
$$\text{For generic } st.$$

However $\operatorname{codim}(\tilde{\Sigma}_{s,t} \cap p^{-1}(g); V_{(g,s,t)}) = \operatorname{codim}(\Sigma_{s,t}, \tilde{V}_{s,t})$ for generic g, s, t. Thus for generic $\tau \in T$ we find $\operatorname{codim}(\mathscr{S}(V_\tau), \tilde{V}_\tau) \geq k$, as desired.

Clearly our theorem can be inductively extended to any sequence $X_1, \cdots, X_r$ of germs of perfect analytic subvarieties of C^n satisfying

$$\operatorname{codim}\left(\bigcap_{i=1}^{t} X_i, C^n\right) = \sum_{i=1}^{t} \operatorname{codim}(X_i, C^n), \quad \text{for all } t \leqq r.$$

We call such a sequence a proper sequence.

If the sequence satisfies the stonger condition

$$\operatorname{codim}\left(\bigcap_{j=1}^{t} X_{i_j}, C^n\right) = \sum_{j=1}^{t} \operatorname{codim}(X_{i_j}, C^n) \text{ of } \{1, \cdots, n\}$$

for all subsequences $i_1 < i_2 \cdots < i_t$ then we shall call it a very proper subsequence.

In the case of germs of determinantal varieties we can obtain

COROLLARY 1. *Let $X_1, \cdots, X_r$ be a proper sequence of germs of determinantal subvarieties of C^n of type (m_i, n_i, l_i) respectively. For each i_j such that X_{i_j} is not a complete intersection, set $k_j = m_{i_j} + n_{i_j} - 2l_{i_j} + 3$. Let $X = \bigcap_{j=1}^{r} X_i$. Then if* $\dim X < \min_j k_j$, *X is smoothable.*

COROLLARY 2. *Let $X_1, \cdots, X_r$ be a proper sequence of germs of analytic subvarieties of C^n. Suppose each X_i is either a complete intersection or a perfect subvariety of codimension 2. Let $X = \bigcap X_i$. Then* $\dim X \leqq 3$ *implies X is smoothable.*

PROOF. By [9], [11] if X_i is perfect of codimension 2 it is determinantal of type $(n, n-1, n-1)$. Thus $k_j = 4$. Now apply the previous corollary.

We now turn to the versal deformation spaces of intersections of the above type.

3. Versal deformation sapces. For a germ of an analytic subvariety X of C^n let Θ_X denote the sheaf of tangent vectors of X. Then the $\mathcal{O}_X$ module of isomorphism classes of first order infinitesimal deformations of X, T^1_X, is defined by the exact sequence

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{C^n}|_X \xrightarrow{\ \rho\ } N_X \longrightarrow T^1_x \longrightarrow 0$$

where $N_x = \operatorname{Hom}_{\mathcal{O}_{C^n}}(\mathscr{I}(X), \mathcal{O}_X)$ and ρ is the mapping taking $\sum_i \theta_i(x) \cdot \partial/\partial x$ to the homomorphism $f_i \mapsto \sum_i \theta_i \cdot \partial_i/\partial x_i$. See [12] for further details.

Now let X_1, X_2 be germs of analytic subvarieties at the origin and set $X = X_1 \cap X_2$. Consider $\mathcal{O}_X$ to be a module over $\mathcal{O}_{X_i}$ and let $\alpha_i: N_{X_i} \otimes \mathcal{O}_{X_i}$

with the variety $\tilde{X}_2$ defined by

$$H(z,t) = z^2 - z^3 + z_1 + z_4 + t_5 z_3^2 + t_6 z_3 + t_7 z_2^2 + t_8 z_2 + t_9 z_2 z_3 + t_{10}.$$

Note that the first four parameters $t_1, \cdots, t_4$ correspond to moving X in X_2 while holding X_2 fixed while the last six parameters correspond to moving X in X_1 holding X_1 fixed. Note also that T^1_X is not the direct sum of $T^1_{X_1}$ and $T^1_{X_2}$ since X_2 being rigid has $T^1_{X_2} = \{0\}$ and $\dim T^1_{X_1} = 4$. Also as deformations of X_2, all the $X_{2,t}$ are isomorphic to X_2 and $\tilde{X}_2 \approx X_2 \times C^{10}$. However these trivial deformations of X_2 induce nontrivial deformations of $X_1 \cap X_2 = X$.

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On the Hessian of the Carathéodory Metric

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Abstract. The generalized lower Hessian of an upper semi-continuous function f near a point z in $\mathbb{C}^n$ is introduced (for $n = 1$ see Heins, Nagoya Math. J. 21 (1962), 1-60). With this we introduce a "sectional curvature" and we prove that the sectional curvature of the Carathéodory-Reiffen metric is always ≤ -4. This generalizes a result of Suita (Kodai Math. Sem. Rep. 25 (1973), 215-218) in the one dimensional case. The sectional curvatures of the ball and polydisk are always -4. Few other properties of the Hessian of the above metric are shown.

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b. Richard Mandelbaum (with M. Schaps).

[1] On the Smoothing and Deformation of Perfect Varieties.

Conclusion

Unfortunately, shortly after this grant was awarded, circumstances led to the separation of the principal investigators. During most of the grant period, they were located on three separate continents and this led to certain difficulties of interaction not foreseen when the grant was undertaken.

In addition, the scope of the proposal was extremely wide and thus, in certain areas, only preliminary results were obtainable. The investigators are thus continuing much of the work begun during the grant period in their own individual research.